

Integrability, Jacobians and Calabi-Yau threefolds¹

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Abstract

The integrable systems associated with Seiberg-Witten geometry are considered both from the Hitchin-Donagi-Witten gauge model and in terms of intermediate Jacobians of Calabi-Yau threefolds. Dual pairs and enhancement of gauge symmetries are discussed on the basis of a map from the Donagi-Witten “moduli” into the moduli of complex structures of the Calabi-Yau threefold.

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Looking for a unified point of view to describe the great amount of exciting new results appearing in the last few months in the context of string theory can be considered at this moment premature and maybe a pure question of taste. Nevertheless, in this lecture we would like to present a conjectural unified approach supported by some evidence [1] and based on the mathematical structure of integrable systems [2, 3, 4].

1 Integrability and Seiberg-Witten geometry

A very elegant way to describe Seiberg-Witten geometry [5, 6] for $N = 2$ non abelian gauge theories, is [3] by means of families of complex tori [4].

Let

$$\pi : X \rightarrow U \quad (1)$$

with fiber X_u ($u \in U$) a complex tori of dimension g . We will assume that X is a symplectic manifold possessing a closed non degenerate holomorphic two form w and that $\dim U$ is equal to g . Given a basis $\gamma_i(u)$ $i = 1, \dots, g$ of $H_1(X_u, \mathbb{Z})$, we define “special coordinates”

$$a_i(u) = \oint_{\gamma_i(u)} \lambda, \quad (2)$$

where

$$w = d\lambda. \quad (3)$$

These special coordinates define hamiltonian vector fields which are tangent to X_u . Denoting H_g the Siegel upper half space of $g \times g$ symmetric complex matrices Z with $\text{Im } Z > 0$, the period map

$$p : U \rightarrow H_g \quad (4)$$

is defined in such a way that the complex tori X_u is given by $C^g / \Lambda_{p(u)}$ where $\Lambda_{p(u)}$ is the lattice generated by $(1, Z_u = p(u))$. The “dual” set of special coordinates is defined by

$$a_i^D(u) = \oint_{\gamma_i^D(u)} \lambda, \quad (5)$$

and the “duality” relations between a_i and a_i^D are determined by the period map (4) as follows [4]

$$da_i(u) = p_{ij}(u) da_j^D(u), \quad (6)$$

which for $g = 1$ gives the well known duality relation [5] for $SU(2)$.

The physical interpretation of the family (1) consist in associating U to the moduli space determined by some flat potential of a $N = 2$ gauge theory, and the fiber X_u , together with equations (2) and (6), with the BPS mass spectrum formula.

2 Hitchin's gauge models and Prym varieties

There exists a way to associate with a given gauge group G and a Riemann surface Σ , a family (1) of complex tori [7, 4, 3]. This procedure is based on the concept of spectral curve.

Let us consider as Σ the elliptic curve E_τ

$$y^2 = (x - e_1(\tau))(x - e_2(\tau))(x - e_3(\tau)), \quad (7)$$

and let ϕ be a Higgs field defined on E_τ as a holomorphic 1-form valued in the adjoint representation of G . The spectral curve C is defined by

$$\det(t - \phi) = 0. \quad (8)$$

Denoting $J(C)$ the Jacobian of C , which for $G = SU(2)$ will have dimension two, we can define the Prym-Tjurin subvariety $P(C, \sigma)$ as

$$P(C, \sigma) = (1 - \sigma)J(C), \quad (9)$$

where σ is the automorphism

$$\sigma : t \rightarrow -t \quad (10)$$

of the curve (8) for ϕ valued in the adjoint of $SU(2)$.

Let us now consider the Higgs invariant polynomial $tr\phi^2$, which is a quadratic differential on E_τ . We can evaluate $tr\phi^2$ on the basis (one dimensional) of holomorphic two differentials of E_τ . Let us denote by u the corresponding coefficient. The family (1) of complex tori for $SU(2)$ is now defined [3] with U parameterizing the different values of $tr\phi^2$, and with the fiber X_u given by the Prym-Tjurin variety $P(C_u, \sigma)$.

It is important here to stress the difference between Hitchin-Donagi-Witten construction and the way to associate spectral curves to Lax-hamiltonians systems [8]. In this second case one starts with a map $A : P^1 \rightarrow \mathcal{G}$ on some Lie algebra \mathcal{G} . The spectral curve C defined by

$$\det(z \mathbf{1} - A(\xi)) = 0 \quad (11)$$

is a sheeted covering $p : C \rightarrow P^1$ with fiber at a point ξ_0 being identified with the set of extremal weights $\{\lambda_1, \dots, \lambda_r\}$ of $h(\xi_0)$ (the centralizer of $A(\xi_0)$). The Weyl group is acting by permutations S_d on this set of weights (see reference [8] for more technical details). In the Hitchin-Donagi-Witten approach one starts on a general reference Riemann surface, reducing for the cases of physical interest to E_τ . The equivalent to $A(\xi)$ in (11) is the Higgs field defined on E_τ , and the way we parameterize this Higgs field is by the coordinates u_i of the invariant polynomials relative to the basis of the corresponding q -holomorphic differentials of the reference surface E_τ .

The interest of working on E_τ is that we can define an alternative family of complex tori keeping fixed the Higgs field while changing the moduli τ of the reference Riemann

surface E_τ . Notice that working on E_τ is not affecting the way the Weyl group is acting on the spectral sets.

3 Intermediate Jacobians for threefolds and dual pairs

In the previous section we have discussed a model for families of abelian varieties based on Hitchin's construction. The ingredients were a reference Riemann surface Σ and a Higgs system defined on Σ with gauge group G . A different geometrical model for families of complex tori can be obtained for Calabi-Yau threefolds using Griffiths intermediate Jacobians [9]. Given a Calabi-Yau threefold Y , let us denote by $J(Y)$ the corresponding intermediate Jacobian (see reference [4] for definitions). The dimension of $J(Y)$ is given by $h_{2,1}(Y)$ and the family of complex tori is obtained by fibering $J(Y)$ on the moduli of complex structures of Y . By means of the intermediate Jacobian $J(Y)$ we map 3-cycles of Y into 1-cycles of $J(Y)$. Recent results in dual pairs [10, 11] strongly motivate us to look for a way to connect the Hitchin model based on the data (E_τ, G) and Griffiths construction based on a Calabi-Yau threefold Y , namely interpreting these two constructions as a sort of dual pair with

$$\begin{aligned} \tau &\rightarrow \text{dilaton} \\ \text{rank } G &\rightarrow h_{2,1}(Y) - 1 \\ \text{1-cycles of } J(C) &\rightarrow \text{3-cycles of } Y. \end{aligned} \tag{12}$$

Relations (12) should be interpreted at this point only as a conjectural framework. Before going into a more detailed discussion let us make some general comments.

- i) Intermediate Jacobians were used in [4] (see also [12]) to define integrable systems. In order to do that it is important to enlarge the Jacobian by adding the holomorphic gauge. Namely, in physics terminology, by including the graviphoton. We will not discuss this important issue in this note.
- ii) Taking into account that we are inspired by the heterotic-type II dual pairs, we should consider Calabi-Yau threefolds that are K_3 -fibrations [13]. For these fibrations we know [14] that the size of the P^1 -basis is connected with the heterotic dilaton. Following (12) we should try to connect this size with the value of τ for the reference surface in the Hitchin-Donagi-Witten construction.
- iii) In the Donagi-Witten approach we have two types of 1-cycles, namely the ones in $J(C)$ coming from the Jacobian of the surface E_τ and the rest. In the next section we will review some of the results of [1] proposing a physical interpretation for both types of one cycles.

4 From Donagi-Witten moduli to a threefold moduli

We will consider the case of $G = SU(2)$ and E_τ as the reference Riemann surface. We parameterize the corresponding spectral curve by (\hat{u}, τ) , where $\hat{u} \equiv u/(\frac{1}{2}m^2)$ and with m fixing the residue of the pole of the Higgs field on E_τ at infinity (see [3]). Depending on what type of 1-cycle degenerates we can differentiate the following two set of singularity loci²

$$\text{type 1: } \begin{cases} \hat{\mathcal{C}}_0 & \equiv \{ \hat{u}(\tau) = \frac{3}{2}e_1(\tau) \} \\ \hat{\mathcal{C}}_C^{(1)} & \equiv \{ \hat{u}(\tau) = e_3(\tau) + \frac{1}{2}e_1(\tau) \}, \\ \hat{\mathcal{C}}_C^{(2)} & \equiv \{ \hat{u}(\tau) = e_2(\tau) + \frac{1}{2}e_1(\tau) \} \end{cases}, \quad \text{type 2: } \begin{cases} \hat{\mathcal{C}}_\infty & \equiv \{ \tau = i\infty / \epsilon = 0 \} \\ \hat{\mathcal{C}}_1^+ & \equiv \{ \tau = 0 / \epsilon = 1 \} \\ \hat{\mathcal{C}}_1^- & \equiv \{ \tau = 1 / \epsilon = -1 \}, \end{cases} \quad (13)$$

with $\epsilon = e^{i\pi\tau}$.

By blowing up the crossing between $\hat{\mathcal{C}}_C^{(1)}$, $\hat{\mathcal{C}}_C^{(2)}$ and $\hat{\mathcal{C}}_\infty$ we get an exceptional divisor E parameterized by $\tilde{u} \equiv \hat{u}/\epsilon$, that we can identify with the $N=2$ Seiberg-Witten quantum moduli for $SU(2)$.

The action of $T : \tau \rightarrow \tau + 1$ on (\tilde{u}, ϵ) -variables is given by

$$\begin{aligned} \tilde{u} &\rightarrow -\tilde{u}, \\ \epsilon &\rightarrow -\epsilon. \end{aligned} \quad (14)$$

If we quotient by the action (14) we get the F_2 -Hirzebruch space

$$\xi\zeta = \eta^2, \quad (15)$$

with

$$\xi \equiv \tilde{u}^2, \quad \eta \equiv \frac{\tilde{u}}{\epsilon}, \quad \zeta \equiv \frac{1}{\epsilon^2}. \quad (16)$$

This is the space we want to relate with the moduli of complex structures of the mirror of WP_{11226}^{12} [15, 16], with $h_{2,1} = 2$, and defining polynomial

$$p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6. \quad (17)$$

Using variables $x = \frac{1}{\phi^2}$ and $y = \frac{-\phi}{864\psi^6}$, the blow up of the tangency between the conifold locus and the weak coupling locus $\{y = 0\}$ produces an exceptional divisor parameterized by $\frac{xy^2}{(1-x)^2}$. Identifying the loci $\hat{\mathcal{C}}_C^{(1)}$ and $\hat{\mathcal{C}}_C^{(2)}$ with the two branches of the conifold, $\hat{\mathcal{C}}_\infty$ with the weak coupling locus $\{y = 0\}$, and identifying the blow ups in both spaces (the F_2 -space (\hat{u}, τ) and the Calabi-Yau moduli), we get

$$\frac{yx^2}{(1-x)^2} = \frac{\epsilon^2}{\hat{u}^2} = \frac{1}{\tilde{u}^2}. \quad (18)$$

²The loci grouped as type 2 imply the contraction of all 1-cycles in $J(C)$ for $SU(2)$, both coming from the reference surface E_τ and the rest.

This is the point particle limit described in [17], in which the string moduli exactly reproduces the quantum moduli space for $N=2$ $SU(2)$ Yang-Mills. Notice also that (18) goes into the direction (12) of interpreting the moduli τ of the reference Riemann surface E_τ as the dilaton. By this identification the contracting 3-cycles associated with conifold like singularities [18] are mapped into 1-cycles singularities of the type 1 (13).

The very symmetric structure of the loci $\hat{\mathcal{C}}_C^{(1)}$, $\hat{\mathcal{C}}_C^{(2)}$ and $\hat{\mathcal{C}}_1^+$, $\hat{\mathcal{C}}_1^-$ in (13), motivates us to try to extend the relation between Donagi-Witten moduli and the threefold moduli beyond the point particle limit.

In order to analyze the singularities of type 2, we should work out the family of abelian varieties obtained by changing the moduli of E_τ while maintaining fixed the (4-dimensional) Higgs field, i.e. the value of u . Recall that u was identified with the coefficient that arises when we expand the invariant polynomial $tr\phi^2$ in the basis of quadratic differentials of E_τ , where ϕ is the Higgs field on E_τ . Denoting by $C_{\tau,u}$ the corresponding spectral curve defined by (8), then we can, for fixed u , define a family of abelian varieties over the moduli of E_τ with fiber the Prym manifold $P(C_{\tau,u})$. At the singular loci of type 2 the 1-cycle of E_τ contracts to a point. For this family of abelian varieties the τ -plane is playing the role of moduli space and the Z_2 centralizer of $Sl(2; Z)$ the one of the classical Weyl group [19]. By the identification of the τ and the heterotic dilaton S , and the relation between S and the y variable [11], it would be natural to map the loci $\hat{\mathcal{C}}_1^+$, $\hat{\mathcal{C}}_1^-$ with the two branches $\{\sqrt{y} = \pm 1\}$ ($\{\phi = \pm 1\}$) of the strong coupling locus $\{y = 1\}$. Moreover, the locus $\{y = 1\}$ corresponds to a wall of the Kähler cone in the moduli space of the mirror Calabi-Yau, which can be interpreted as a topology changing singularity with non perturbative enhancement of non-abelian gauge symmetry [20, 21, 22]. In reference [20] this enhancement of symmetry was determined by the type of ALE space describing the singular Calabi-Yau manifold, being $G' = SU(2)$ for WP_{11226}^{12} . For the family of abelian varieties over the τ -plane we have described above, the monodromies around the singular loci of type 2 are, up to classical Weyl monodromies in the centralizer of $Sl(2; Z)$, conjugated to T^2 , in preliminary agreement with the results of [20]³.

A tentative map

$$x = \frac{3/2e_1(\tau)}{3/2e_1(\tau) - \hat{u}}, \quad \sqrt{y} = -\frac{e_2(\tau) - e_3(\tau)}{3e_1(\tau)}, \quad (19)$$

between the (\hat{u}, τ) -plane and the Calabi-Yau threefold moduli was proposed in [1]. By this map, the correspondence between the conifold locus and $\hat{\mathcal{C}}_C^{(1)}$, $\hat{\mathcal{C}}_C^{(2)}$, and $\{y = 1\}$ and $\hat{\mathcal{C}}_1^+$, $\hat{\mathcal{C}}_1^-$ described above are satisfied [23]. The map (19) reproduces the correct point particle limit of [17].

The pull back by this map of the complex tori defined by the intermediate Jacobian of the Calabi-Yau should map the 3-cycles in $H_3(Y)$ related with conifold singularities to

³In this general discussion we can only compare monodromies that in [20] depend on the genus g of the curve of singularities, which determines the number of hypermultiplets in the adjoint of G' that also acquire zero mass at the enhancement of symmetry locus.

1-cycles of $J(C)$ which are not coming from E_τ , and the 3-cycles in $H_3(Y)$ related to the non-perturbative enhancement of non-abelian gauge symmetries to the part of the $J(C)$ coming from the reference surface E_τ

$$\begin{aligned} \pi : J(C) &\rightarrow J(E_\tau) \\ \text{Ker } \pi &\leftrightarrow \text{enhancement of symmetry 3-cycles} \end{aligned} \tag{20}$$

To finish let us just put the question on the possible (F-theoretical) physical meaning of models defined on reference Riemann surfaces with genus bigger than one.

5 Final Remarks

There are two things we want to stress. The first one is connected with duality. To keep alive the τ parameter, as it is natural to do in the Donagi-Witten framework, allows to define the action of the $Sl(2; Z)$ duality transformations in a very natural way as the modular transformations on τ , and to induce them by the map (19) into the Calabi-Yau moduli.

The second is related with the crucial role of integrability. The abstract structure of a moduli manifold fibered by complex tori defined by the lattice of BPS-particles, admits two interesting mathematical models. The one of Hitchin related to gauge theories on a Riemann surface, and the one defined using intermediate Jacobians related to Calabi-Yau threefolds. It would be appealing if that general framework could help in a unified understanding of dual pairs, string duality and non perturbative enhancements of gauge symmetry. The previous lecture was a modest attempt of that.

Note: As we were finishing this note it appeared the paper [24], where the question of integrability and Calabi-Yau K_3 -fibrations is considered.

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References

- [1] C. Gómez, R. Hernández and E. López, “ S -Duality and the Calabi-Yau Interpretation of the $N=4$ to $N=2$ Flow”, hep-th/9512017.
- [2] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. **B355** (1995) 466, hep-th/9505035.
 E. Martinec , N. Warner, “Integrable Systems and Supersymmetric Gauge Theory”, hep-th/9509161.
 E. Martinec, “Integrable Structures in Supersymmetric Gauge and String Theory”, hep-th/9510204.
 E. Martinec , N.P. Warner, “Integrability in $N=2$ Gauge Theory: A Proof”, hep-th/9511052.
 H. Itoyama , A. Morozov, “Integrability and Seiberg-Witten Theory: Curves and Periods”, hep-th/9511126.
 H. Itoyama , A. Morozov, “Prepotential and the Seiberg-Witten Theory”, hep-th/9512161.
 A. Gorsky , A. Marshakov , A. Mironov , A. Morozov, “ $N=2$ Supersymmetric QCD and Integrable Spin Chains: Rational Case $N_f < 2N_c$ ”, hep-th/9603140.
- [3] R. Donagi and E. Witten, “Supersymmetric Yang-Mills and Integrable Systems”, hep-th/9510101.
- [4] R. Donagi and E. Markman, “Spectral Curves, Algebraically Completely Integrable Hamiltonians Systems, and Moduli of Bundles”, alg-geom/9507017.
- [5] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19, hep-th/9407087.
- [6] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484, hep-th/9408099.
- [7] N. Hitchin, Duke Math. J. **54** (1987) 91.
- [8] V. Kanev, in Proc. of Symposia in Pure Mathematics, Vol. 49.
- [9] See H. Clemens and P. Griffiths, Ann. of Math. **95** (1972) 281.
- [10] C. Hull and P. Townsend, Nucl. Phys. **B438** (1995) 109, hep-th/9410167.
 E. Witten, Nucl. Phys. **B443** (1995) 85, hep-th/9503124.
 A. Sen, Nucl. Phys. **B450** (1995) 103, hep-th/9504027.
 J. Harvey and A. Strominger, Nucl. Phys. **B449** (1995) 535, hep-th/9504047.
 S. Ferrara , J. A. Harvey , A. Strominger , C. Vafa, “Second-Quantized Mirror Symmetry”, hep-th/9505162.

- [11] S. Kachru and C. Vafa, Nucl. Phys. **B450** (1995) 69, hep-th/9505105.
- [12] E. Witten, “Quantum Background Independence In String Theory”, hep-th/9306122.
- [13] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. **357** (1995) 313, hep-th/9506112.
- [14] P. S. Aspinwall and J. Louis, “On the Ubiquity of $K3$ Fibrations in String Duality”, hep-th/9510234.
- [15] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. **B416** (1994) 481, hep-th/9308083.
- [16] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, Comm. Math. Phys. **167** (1995) 301, hep-th/9308122.
- [17] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. **B459** (1996) 537, hep-th/9508155.
- [18] A. Strominger, Nucl. Phys. **B451** (1995) 96, hep-th/9504090.
- [19] C. Gómez, R. Hernández and E. López, work in progress.
- [20] S. Katz, D. R. Morrison and M. R. Plesser, “Enhanced Gauge Symmetry in Type II String Theory”, hep-th/9601108.
- [21] M. Bershadsky, V. Sadov and C. Vafa, “D-Strings on D-Manifolds”, hep-th/9510225.
- [22] A. Klemm and P. Mayr, “Strong Coupling Singularities and Non-abelian Gauge Symmetries in $N = 2$ String Theory”, hep-th/9601014.
- [23] C. Gómez, R. Hernández and E. López, “Integrability, Duality and Strings”, hep-th/9604043.
- [24] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, “Self-Dual Strings and $N = 2$ Supersymmetric Field Theory”, hep-th/9604034.